Q:→ Charpit's method (General method of solving partial differential equations of order one but of any degree)

Let the given partial differential of first-order & non-linear in p & q be.

$$f(x, y, z, p, q) = 0 \longrightarrow (1)$$

We know that

$$dz = p\,dx + q\,dy \longrightarrow (2)$$

The next step consists in finding another relation

$$F(x, y, z, p, q) = 0 \longrightarrow (3)$$

Such that when the values of p & q obtained by solving (1) & (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiate partially (1) & (3) w.r.t x & y & get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \longrightarrow (4)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \longrightarrow (5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial y} = 0 \longrightarrow (6)$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \longrightarrow (7)$$

Eliminating $\frac{\partial p}{\partial x}$ from (4) & (5), we get

$$\left(\frac{\partial f}{\partial x} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial x} \cdot \frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial z} \frac{\partial f}{\partial p}\right) p$$

$$+ \left(\frac{\partial f}{\partial q} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial q} \cdot \frac{\partial f}{\partial p}\right) \frac{\partial q}{\partial x} = 0 \longrightarrow (8)$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from 6 & 7, we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q +$$

$$\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \longrightarrow 9$$

$\because \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$, the last term in 8 is the same as that in 9, except for a minus sign & hence they cancel on adding 8 & 9.

Therefore adding 8 & 9 and rearranging the terms we obtain

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\cdot\frac{\partial F}{\partial z}$$

$$+ \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \longrightarrow 10$$

then auxillary eqn is –

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or

$$\boxed{\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}}$$

Q3. → Apply Charpit's method the find the complete integral of $yzp^2 - q = 0$

Sol^n: → Here $f(x,y,z,p,q) = yzp^2 - q \longrightarrow (1)$

Charpit's auxillary equ^n are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{0 + p(yp^2)} = \frac{dq}{zp^2 + q(yp^2)} = \frac{dz}{-p\cdot(2pyz) - q\cdot(-1)} = \frac{dx}{-2yzp}$$

$$= \frac{dy}{-(-1)} \longrightarrow (2)$$

Taking the first & fifth fraction,

$$\frac{dp}{yp^3} = \frac{dy}{1}$$

$$p^{-3}dp = y\,dy$$

$$\frac{p^{-2}}{-2} = \frac{y^2}{2} + \frac{c_1^2}{2} \quad \left\{\text{where } \frac{c_1^2}{2} \text{ is constant of integration}\right\}$$

$$p^{-2} = c_1^2 - y^2$$

$$p = \frac{1}{(c_1^2 - y^2)^{\frac{1}{2}}} \longrightarrow (3)$$

By equ^n (1)

$$q = yzp^2$$

put the value of p from equ^n (3)

$$q = \frac{yz}{c_1^2 - y^2} \longrightarrow (4)$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{dx}{(c_1^2 - y^2)^{\frac{1}{2}}} + \frac{yz\,dy}{(c_1^2 + y^2)}$$

$$(c_1^2 - y^2)^{\frac{1}{2}}\,dz = dx + \frac{yz\,dy}{(c_1^2 - y^2)^{\frac{1}{2}}}$$

$$(c_1^2 - y^2)^{\frac{1}{2}}\,dz - \frac{yz\,dy}{(c_1^2 - y^2)^{\frac{1}{2}}} = dx$$

$$d\left[z\,(c_1^2 - y^2)^{\frac{1}{2}}\right] = dx$$

Integrating above eqn

$$z\,(c_1^2 - y^2)^{\frac{1}{2}} = x + c_2$$

squaring both the sides

$$\boxed{z^2\,(c_1^2 - y^2) = (x + c_2)^2}$$

Where c_1 & c_2 are constants.